1. Recap from last class.

Definition: The *definite integral of $f(x)$ from a to b*, denoted

$$\int_a^b f(x)\,dx,$$

is the area under the curve of f from a to b.

2. Basic properties of definite integrals.

Fill in the following rules for definite integrals by interpreting the integrals as areas, or by thinking about the Riemann sums which approximate the integrals. (Below, k is a constant.)

(a) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \text{(by definition)}$$

(b) $$\int_a^a f(x)\,dx =$$

(c) $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$$

(d) $$\int_a^b k\,dx =$$

(e) $$\int_a^b k \cdot f(x)\,dx =$$

(f) $$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx =$$

3. More properties.

(a) If $f(x) \geq g(x)$ for all x in $[a, b]$, what is the relationship between $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$?

Adapted from work of Jun Le Goh

Definition: Let f be an integrable function on $[a,b]$. The *average value of f on $[a,b]$* is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

(b) Let $\min f$ be the global minimum of f on $[a,b]$. What is the relationship between $\int_a^b f(x)\,dx$ and $\int_a^b \min f\,dx$?

(c) What is the relationship between the average value of f on $[a,b]$ and $\min f$?

(d) Similarly, what is the relationship between the average value of f on $[a,b]$ and $\max f$?

(e) Use the above inequalities to prove the Mean Value Theorem for Definite Integrals: if f is continuous on $[a,b]$, then there is some c in $[a,b]$ such that $f(c)$ equals the average value of f on $[a,b]$.
(*Hint: use EVT and IVT.*)

4. Extra practice.

So far, we've been pretending that functions are always *integrable*, i.e., their definite integrals always exist.

> **Fact:** Every continuous function is integrable. In fact, every piecewise continuous function is integrable. Not all functions are integrable, however.

Consider the following function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) What are the upper and lower sums of f over $[0,1]$? Why?
(*Hint: In every open interval* (a,b) *there is a rational number and an irrational number.*)

(b) Use that to show that f is not integrable over $[0,1]$.